Math 246C Lecture 1 Notes

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1 Introduction to Riemann Surfaces

In this course, we will study two main topics:

1. Introduction to Riemann surfaces.

2. Introduction to several complex variables.

1.1 Complex charts and atlases

Definition 1.1. Let X be a Hasudorff topological space. A **complex chart** on X is a homeomorphism $\varphi : U \to V$, where $U \subseteq X$ and $V \subseteq \mathbb{C}$ are open. Two charts $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ are called **compatible** if $U_1 \cap U_2 = \varnothing$ or the **transition map** $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is holomorphic. A **complex atlas** on X is a collection of pairwise compatible charts $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in A}$ such that $X = \bigcup_{\alpha \in A} U_\alpha$.

Remark 1.1. It follows that $\varphi_2 \circ \varphi_1^{-1}$ is a holomorphic diffeomorphism.

Proposition 1.1. *Let $\mathscr{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha\}$ be a complex atlas for X. The collection $\widehat{\mathscr{A}} = \{\varphi : U \to V : \varphi$ is a chart on X, φ and φ_α are compatible $\forall \alpha\}$ is a complex atlas for X, $\mathscr{A} \subseteq \widehat{\mathscr{A}}$, and this atlas is maximal. If $\mathscr{A} \subseteq \mathscr{B}$, then $\mathscr{B} \subseteq \widehat{\mathscr{A}}$*

Proof. We only need to check that $\widehat{\mathscr{A}}$ is an atlas. Let $\varphi_1 : U_1 \to V_1$, $\varphi_2 : U_2 \to V_2$ be charts in $\widehat{\mathscr{A}}$, and check that $\varphi_2 \circ \varphi_1^{-1}$ is holomorphic: Let $z \in \varphi_1(U_1 \cap U_2)$ and let $\varphi_\alpha : U_\alpha \to V_\alpha$ be a chart in $\mathscr{A}$ such that $\varphi_1^{-1}(z) \in U_\alpha$. Then $\varphi_1(U_1 \cap U_2 \cap U_\alpha)$ is a neighborhood of z, and $\varphi_2 \circ \varphi_1^{-1}$:

$$\varphi_1(U_1 \cap U_2 \cap U_\alpha) \xrightarrow{\varphi_\alpha \circ \varphi_1^{-1}} \varphi_\alpha(U_1 \cap U_2 \cap U_\alpha) \xrightarrow{\varphi_2 \circ \varphi_\alpha^{-1}} \varphi_2(U_1 \cap U_2 \cap U_\alpha)$$

is holomorphic. $\square$

Remark 1.2. An atlas of the form $\widehat{\mathscr{A}}$ is called **maximal**.

Definition 1.2. We say that atlases $\mathscr{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha\}, \mathscr{B} = \{\varphi'_\beta : U'_\beta \to V'_\beta\}$ are **equivalent** if $\varphi_\alpha, \varphi'_\beta$ are compatible for all α, β.

Remark 1.3. $\mathscr{A}$ is equivalent to $\mathscr{B}$ iff $\widehat{\mathscr{A}} = \widehat{\mathscr{B}}$.

1.2 Riemann surfaces

Definition 1.3. A **complex structure** on X is given by a maximal atlas on X. A **Riemann surface** is a connected, Hausdorff topological space equipped with a complex structure.

Example 1.1. Let $\Omega \subseteq \mathbb{C}$ be open and connected. Then Ω is a Riemann surface when equipped with the atlas $\{1 : \Omega \to \Omega\}$.

Example 1.2. The Riemann sphere $\widehat{\mathbb{C}} \cup \{\infty\}$ with the usual topology is a Riemann surface. Let $U_1 = \mathbb{C}$, $U_2 = \widehat{\mathbb{C}} \setminus \{0\}$ be open, and define the charts $\varphi_1 : U_1 \to \mathbb{C}$ sending $z \mapsto z$ and $\varphi_2 : U_2 \to \mathbb{C}$ send

$$\varphi_2(z) = \begin{cases} 1/z & z \in \mathbb{C} \setminus \{0\} \\ 0 & z = \infty. \end{cases}$$

To check compatibility, $\varphi_2 \circ \varphi_1^{-1}(z) = 1/z$ as a function from $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$. The atlas $(\varphi_j, U_j)_{j=1,2}$ gives rise to a Riemann surface structure on $\widehat{\mathbb{C}}$.

Example 1.3 (complex tori)**.** Let $e_1, e_2 \in \mathbb{C}$ be $\mathbb{R}$-linearly independent, and let Λ be the lattice $\Lambda = \{me_1 + ne_2 : m, n \in \mathbb{Z}\} \subseteq \mathbb{C}$. We have the equivalence relation $z \sim w$ if $z - w \in \Lambda$ and let $\mathbb{C}/\Lambda = z + \Lambda : z \in \mathbb{C}\}$ be the collection of equivalence classes. We have the projection map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ sending $z \mapsto z + \Lambda$. We equip $\mathbb{C}/\Lambda$ with the strongest topology such that π is continuous: $O \subseteq \mathbb{C}/\Lambda$ is open if $\pi^{-1}(O) \subseteq \mathbb{C}$ is open. Then $\mathbb{C}/\Lambda$ is connected and compact. Compactness follows from $\mathbb{C}/\Lambda = \pi(\{te_1 + se_2 : 0 \le t, s \le 1\})$.

We claim that π is an open map. Let $V \subseteq \mathbb{C}$ be open. Then $\pi(V) \subseteq \mathbb{C}/\Lambda$ is open iff $\pi^{-1}(\pi(V)) \subseteq \mathbb{C}$ is open. This is $\pi^{-1}(\pi(V)) = \{z \in \mathbb{C} : \pi(z) \in \pi(V)\} = \bigcup_{\zeta \in \Lambda}(\zeta + V)$.

We need complex charts on $\mathbb{C}/\Lambda$: Let $V \subseteq \mathbb{C}$ be open such that no 2 distinct points of V are equivalent under Λ. Then $\pi|_V : V \to \pi(V) = U$ is a homeomorphism, and $\varphi = (\pi_V)^{-1}$ is a chart.